ON THE CONVERGENCE OF THE BUBNOV-GALERKIN METHOD

(O SKHODIMOSTI METODA BUBNOVA-GALERKINA)

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The convergence of the Bubnov-Galerkin method is studied for linear equations in Hilbert spaces. A new proof of the convergence of this method is given for equations with completely continuous operators. This proof makes it possible to generalize certain known results.

In a separable Hilbert space H, let a linear (possibly unbounded) operator L be given. The domain of definition and the range of values of L are each dense in H. The problem is to solve the equation

$$Lx = f, \qquad f \in H$$

The Bubnov-Galerkin method consists of the following. Let $R_n (n = 1, 2, \ldots)$ be a sequence of finite-dimensional subspaces from the domain of definition of L, and let P_m be orthogonal projection operators on these subspaces. The sequence R_n is assumed to be projectionally complete [1] in H. We are seeking an approximate solution $x_n \in R_m$ of the equation

$$P_n L x_n = P_n f$$

We shall use the notation

$$\tau_n = \min \frac{|P_n L x_n|}{|L x_n|}, \qquad x_n \in R_n$$

In [1] it was proved that if the sequences of subspaces R_n and $L_n = LR_n$, $(n = 1, 2, \ldots)$ are projectionally complete in H then a necessary and sufficient condition for the strong convergence of Lx_n to $f (f \in H)$ when n goes to infinity is that τ be greater than zero, where $\tau = \inf \lim \tau_n$ when $n \to \infty$. Just as in [1], we shall say that the operator L is regular if any two projectionally complete sequences R_n and $L_n = LR_n$ satisfy the condition $\tau > 0$. (We note that for a bounded operator L the projectional completeness of the sequence L_n is implied by the projectional completeness of the sequence R_n.)

Theorem 1. Let a linear bounded operator L have an inverse operator L^{-1}. If there exists a positive number δ such that for every weakly convergent (to zero) sequence $x_n \in H$ with the property $|x_n| = 1$ ($n = 1, 2, \ldots$) and the inequality $\delta \leqslant \inf \lim |(Lx_n, x_n)|$ when $n \to \infty$, then the operator L is regular.

Here, and in what follows, we assume that the bounded linear operators are defined in all of H.

Proof. Let us assume that the operator L is not regular. Then there must exist a projectionally complete sequence of subspaces R_n and a sequence $x_n \in R$ ($n = 1, 2, \ldots$) such that the following relations hold:

$$\lim_{n\to\infty} \frac{|P_nLx_n|}{|Lx_n|} = 0, \quad \text{or} \quad \lim_{n\to\infty} \frac{|P_nLx_n|}{|x_n|} = 0 \tag{1}$$

because $|Lx_n| \leqslant \|L\|\,|x|$. Obviously we may assume that $|x_n| = 1$. Let z be an arbitrary element in H. We have

$$|(x_n, L^*z)| = |(Lx_n, z)| \leqslant |(Lx_n, P_nz)| + |(Lx_n, z - P_nz)| \leqslant$$
$$\leqslant |P_nLx_n|\,|z| + |(Lx_n, z - P_nz)|$$

If we let $n \to \infty$, it follows from (1) that $\lim |P_nLx_n| = 0$, and from the boundedness of the sequence $\{Lx_n\}$ and the projectional completeness of the sequence $\{R_n\}$ it follows that $\lim |(Lx_n, z - P_nz)| = 0$.

Hence, $\lim (x_n, L^*z) = 0$. The range of values of the operator L^* is dense in H, for otherwise there would exist a non-zero $x \in H$ which would be orthogonal to all elements of the form L^*z. This would imply the equation $Lx = 0$, which contradicts the hypothesis that L^{-1} exists. Therefore, the sequence $\{x_n\}$ converges weakly to zero. Furthermore

$$|(Lx_n, x_n)| \leqslant |P_nLx_n|$$

From equation (1) it follows that $\lim |(Lx_n, x_n)| = 0$ as $n \to \infty$. But this contradicts the hypotheses of the theorem. The theorem has thus been proved.

Simultaneously we have also proved that the operator L^{-1} is bounded. Indeed, if the operator L^{-1} were not bounded then there would exist a sequence $x_n \in H$ ($n = 1, 2, \ldots$) for which

$$\lim_{n\to\infty} \frac{|Lx_n|}{|x_n|} = 0$$

Repeating the arguments, which followed directly after (1), with only slight modifications, we would again come to a contradiction.

Theorem 2. Let a linear bounded operator L have the form $L = L_0 + T$, where L_0 is regular, while T is a completely continuous operator. If zero does not belong to the spectrum of the operator L, then the operator L is regular.

Proof. Let us assume that the theorem is false. Then, repeating the proof of Theorem 1, we find that there exists a projectionally complete sequence $\{x_n\}$, $x_n \in R_n$, $|x_n| = 1$, such that

$$\lim_{n\to\infty} |P_n L x_n| = 0 \tag{2}$$

From the existence and boundedness of the operator L^{-1} it follows that

$$|Lx_n| \geqslant \frac{1}{\|L^{-1}\|} \tag{3}$$

By the triangular inequality we have

$$|P_n L_0 x_n| \leqslant |P_n L x_n| + |P_n T x_n|, \qquad |L_0 x_n| \geqslant |Lx_n| - |Tx_n| \tag{4}$$

From known properties of completely continuous operators it follows that

$$\lim_{n\to\infty} |Tx_n| = 0 \tag{5}$$

Making use of inequalities (3) and (4) and equations (2) and (5), we obtain

$$\lim_{n\to\infty} \frac{|P_n L_0 x_n|}{|L_0 x_n|} = 0$$

We have arrived at a contradiction because the operator L_0 is regular by hypothesis. This proves the theorem. In particular cases, when the operator L_0 is the identity or a positive definite operator, this theorem has been known before. Pol'skii has conjectured [1] that the following theorem is true.

Theorem 3. Let L_0 be a linear bounded operator which satisfies the inequality $\inf |(L_0 x, x)| > 0$, $x \in H$, $|x| = 1$, and let T be a completely continuous linear operator. If zero does not belong to the spectrum of the operator $L = L_0 + T$, then the operator L is regular.

Theorem 3 is a direct consequence of Theorem 2 because L_0 is a regular operator [1]. It can be derived quite easily from Theorem 1 also.

Let us now consider the problem of characteristic values. Suppose that

$$L = A + \lambda B$$

where A and B are linear bounded operators and λ is a number in some region D of the complex plane. In the sequel we shall say that λ is a regular point of the operator L if there exists a bounded operator L^{-1} defined in the entire space H; if λ is not a regular point we shall say that λ belongs to the spectrum of the operator L. The values of λ for which the equation $Lx = 0$ has a nontrivial solution, we shall call the characteristic values of L, and this nontrivial solution we shall call a characteristic vector of the operator L belonging to the characteristic number λ. The same statements apply to the operator P_nL which will assumed to be operating only on the elements of R_n.

Theorem 4. Suppose that some closed bounded set $D_0 \subset D$ does not contain any point of the spectrum of the operator L. If the operator L is regular for every $\lambda \in D_0$ then there exists an n_0 such that for all $n > n_0$, D_0 will not contain any point of the spectrum P_nL.

Proof. Let us write

$$\mu_n(\lambda) = \min \frac{|P_nLx_n|}{|x_n|}, \qquad x_n \in R_n$$

It is obvious that $\mu_n(\lambda) = 0$ if and only if λ is a characteristic value of the operator P_nL, and if $\mu_n(\lambda) \neq 0$, then λ is a regular point of the operator P_nL, and

$$\|(P_nL)^{-1}\| = \frac{1}{\mu_n(\lambda)} \tag{6}$$

From the inequality $|x_n| \leqslant \|L^{-1}\| \, |Lx_n|$ we obtain

$$\frac{|P_nLx_n|}{|x_n|} \geqslant \frac{|P_nLx_n|}{\|L^{-1}\| \, |Lx_n|} \geqslant \frac{\tau_n}{\|L^{-1}\|}$$

and, hence

$$\mu_n(\lambda) \geqslant \frac{\tau_n(\lambda)}{\|L^{-1}\|} \tag{7}$$

Since the operator L is regular if $\lambda \in D_0$, it follows from inequality (7) that

$$0 < \mu(\lambda) = \inf \lim \mu_n(\lambda) \quad \text{when } n \to \infty \tag{8}$$

For any $\lambda \in D$ and $\lambda' \in D$, we have

$$\left| \frac{|P_nL(\lambda')x_n|}{|x_n|} - \frac{|P_nL(\lambda)x_n|}{|x_n|} \right| \leqslant \frac{|P_nL(\lambda')x_n - P_nL(\lambda)x_n|}{|x_n|} \leqslant \|B\| \, |\lambda' - \lambda|$$

From this we obtain

$$|\mu_n(\lambda') - \mu_n(\lambda)| \leqslant \|B\| |\lambda' - \lambda| \tag{9}$$

Because of inequalities (8) and (9), for every point $\lambda \in D_0$ there exists a neighborhood such that for all n after a certain one, the inequality $\mu_n(\lambda') \geqslant \mu(\lambda)/2 > 0$ holds for all λ' in this neighborhood. From a well-known lemma on the existence of a finite subcovering, it now follows that there exists a positive number δ such that for all n after a certain one $\mu_n(\lambda) \geqslant \delta$ for all $\lambda \in D_0$. This completes the proof of the theorem.

Theorem 5. Let $\lambda_0 \in D$ be a characteristic value of the operator L such that there exists a neighborhood, of the point λ_0, which does not contain any other points of the spectrum of the operator L, and the operator L is regular for all points $\lambda \neq \lambda_0$ of this neighborhood. Then there exists a sequence $\{\lambda_n\}$ of characteristic values of the operator P_nL which converges to λ_0.

Proof. Inside the given neighborhood we take a circle C with center at λ_0. Since the circle is a closed bounded set, it follows from the proof of Theorem 4 that there exists a positive number δ such that $\mu_n(\lambda) \geqslant \delta$ ($\lambda \in C$) for all n sufficiently large. Let x be one of the characteristic vectors (of the operator L) belonging to the characteristic value λ_0. It is obvious that

$$\lim \frac{|P_nLP_nx|}{|P_nx|} = \frac{|Lx|}{|x|} = 0 \text{ when } n \to \infty$$

and, hence, $\lim \mu_n(\lambda_0) = 0$. Therefore, the following inequalities will be satisfied simultaneously for all n greater than a certain one:

$$\mu_n(\lambda) \geqslant \delta \text{ when } \lambda \in C, \qquad \mu_n(\lambda_0) < \delta$$

hence, the function $\mu_n(\lambda)$ has a minimum at some point λ_n, $|\lambda_n - \lambda_0| < \rho$, where ρ is the radius of the circle C. We will prove that $\mu_n(\lambda_n) = 0$. Let us assume that $\mu_n(\lambda_n)$ is not zero. Then all points of circle $|\lambda_n - \lambda_0| \leqslant \rho$ will be regular points of the operator P_nL, and, therefore, the operator $(P_nL)^{-1}$ will be a holomorphic function of λ in the circle $|\lambda_n - \lambda_0| \leqslant \rho$. But then the norm of the operator $(P_nL)^{-1}$ cannot have a maximum in the circle $|\lambda_n - \lambda_0| \leqslant \rho$. (This fact follows from the theorem of the mean in the same way that the principle of the maximum of the modulus of an analytic function is a consequence of this theorem [2].) From this and from equation (6) it follows that the function $\mu_n(\lambda)$ can not have a minimum different from zero in the circle $|\lambda_n - \lambda_0| \leqslant \rho$. Hence, $\mu_n(\lambda_n) = 0$. By Theorem 4 we have $\lim \lambda_n = 0$ when $n \to \infty$.

Note 1. Theorems 4 and 5 remain true when the operator L is a holomorphic function of λ; the proof hardly differs from the given one.

Note 2. We have considered only bounded operators. The established theorems can be used in the investigation of the convergence of the Bubnov-Galerkin method for unbounded operators if one can introduce a new space H_0 in such a way that the equations of the Bubnov-Galerkin method in H coincide with the equations of the Bubnov-Galerkin method in H_0 for an equation which is equivalent to the original one, but which now has a bounded operator (in H_0) which satisfies the hypotheses of the theorems. The introduction of the new operator is given in [3] for some differential operators.

Note 3. In [1] there was considered a projection method more general than the Bubnov-Galerkin one. In this method one selects two projectionally complete sequences of subspaces $\{R_n\}$ and $\{M_n\}$ in H_1 and H_2, respectively, where H_1 is a Hilbert space which contains the domain of definition of the given operator, and H_2 is a Hilbert space that contains the range of values of this operator. Theorem 2 becomes valid in this case if one changes the formulation as follows.

Let the bounded linear operator L, which has a bounded inverse L^{-1}, have the form $L = L_0 + T$, where T is completely continuous, and L_0 is an operator for which subsequences $\{R_n\}$ and $\{M_n\}$ satisfy condition (A). Then sequences $\{R_n\}$ and $\{M_n\}$ satisfy condition (A) for the operator L also.

Theorems 4 and 5 also remain true if they are reformulated in a corresponding manner. The changes which will have to be made in the proofs are obvious.

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